Research Article

A New Method for Solving Interval and Fuzzy Quadratic Equations of Dual Form

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Abstract

In this paper, a numerical method for solving a quadratic interval equation of dual form is presented. The idea for the method is based on the generalized procedure of interval extension called "interval extended zero" method. It is stated that the solution of interval quadratic equation based on the presented method may be naturally treated as a fuzzy number. An important advantage of the proposed approach is that it substantially decreases the excess width defect. Three numerical examples are included to demonstrate the applicability and validity of the presented method.

Keywords: Extended Zero, Dual Equation, Interval Arithmetic, Fuzzy Equations, Uncertainty Theory.

1. Introduction

In 1975-1978, for first time the authors of (Zadeh, 1975) and (Dubios & Prade, 1978) introduced the concept of fuzzy number and fuzzy arithmetic operations. The reader can find more information on fuzzy numbers and fuzzy arithmetic in (Kaufman & Gupta, 1985) and (Zadeh, 1965). Now, there are many inherent application of fuzzy systems to study a variety of problems such as mathematics, engineering (Chen & Yang, 2000), finance (Buckley, 1987), (Calzi, 1990), economics (Buckley, 1992), (Wu & Chang, 2003) and social sciences which polynomials play a major role in those areas.

One of the important applications of fuzzy number arithmetic is handling fuzzy linear systems and fully fuzzy linear systems. In the studies (Abbasbandy & Ezzati ,2006), (Abbasbandy & Otadi, 2006), (Ferreira *et al.*, 2005), (Mosleh & Otadi, 2010) and (Shieh, 2008) there are some efforts to find the numerical solution of fuzzy polynomial equation such as

$$a_1x + a_2x_2 + \cdots + a_nx_n = a_0,$$

such that $a_0, a_1, \dots, a_n$ are fuzzy numbers. In (Allahviranloo *et al.*, 2007), the authors also presented a method for solving fuzzy nonlinear equations. Furthermore, the authors of (Muzziloi & Reynaerts, 2006) studied the fully fuzzy linear systems of the form

$$A_1x + b_1 = A_2x + b_2,$$

such that A_1, A_2 are square matrices of fuzzy coefficients and b_1, b_2 are fuzzy number vectors. In (Dehghan *et al.*, 2007), the iterative methods are utilized for the fully fuzzy linear systems of the form $Ax = b$ which A is a fuzzy matrix and b is a fuzzy vector.

Additionally, based on a new arithmetic calculation, Mosleh (Mosleh *et al.*, 2008) studied the general duality fully fuzzy linear systems of the form

$$Ax + b = Cx + d,$$

where A and C are square matrices of fuzzy coefficients and b, d are fuzzy numbers.

In 2009, a new method to the solution of interval and fuzzy equations based on the generalized procedure of the interval and fuzzy extension is proposed in (Seastjanov & Dymova, 2009). The key idea is the treatment of the interval zero as an interval centered around zero. It is shown that such proposition is not of heuristic nature, but is the direct consequence of the interval subtraction and division operations.

The authors in (Movahedian *et al.*, 2009) applied this method for solving the linear interval equation in dual form

$$ax = bx + c,$$

where a, b, c are intervals.

The restricted variation method and optimization method have been used for solving the fully fuzzy equations of the form $\tilde{A}\tilde{X}^2 + \tilde{B}\tilde{X} + \tilde{C} = \tilde{D}$ in (Moazam, 2016) and (Allahviranloo & Moazam, 2014), respectively. In (Landoweski, 2017), the RDM interval method is utilized for solving quadratic interval equation. The interval extended zero method is applied to the solution of nonlinear interval and fuzzy equations which includes the form $ax^2 + bx = c$ in (Dymova & Seastjanov, 2018).

Since in the case of the α-cut presentation, fuzzy arithmetic is based on the interval arithmetic rules, the basic definitions of applied interval analysis should also be presented. One of the most inconvenient features of interval arithmetic is the fast increasing of width of intervals obtained as the results of interval calculations. To reduce this undesirable effect, several different modifications of interval arithmetic were proposed.

In this paper, the idea of interval extended zero method is applied for solving the quadratic equation of the dual form

$$ax^2 + bx + c = dx^2 + ex + f,$$

where the coefficients are interval numbers.

Usually, there is no additive inverse element for an arbitrary fuzzy number $u \in E$ (Lai & Hwang, 1992), i.e., there exists no element $v \in E$ such that $u + v = 0$, so it is assumed that two equations

$$ax^2 + bx + c = dx^2 + ex + f,$$

and

$$ax^2 + bx + c - dx^2 - ex - f = 0,$$

are not equivalent, therefore the term 0 in the right hand side of the last equation as is taken an interval centered at zero and then can deal with these two equations as equivalent ones.

The key to the suggested scheme is the observation that the zero value in interval discussion does not mean "nothing" and a more natural is the handling of the interval zero as an interval centred around zero. This assumption allows us to avoid formally what we have as "interval equation's right hand side problem" and to use the classical technique to the solution of the quadratic equations of the dual form.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminaries are needed in subsequent development. The suggested method for solving the quadratic equation of the dual form is presented in section 3. In section 4, three numerical examples are considered to demonstrate the performance of the presented method. Finally, the paper ends with conclusion in section 5.

2. Preliminaries

This section consists of the problem statement, some basic definitions and properties in fuzzy arithmetic and the base of the interval extended zero method.

2.1. Problem statement

Consider the equation

$$ax^2 + bx + c = dx^2 + ex + f.$$

This equation is an arbitrary duality form of quadratic equation, where

$$a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}], c = [\underline{c}, \overline{c}], d = [\underline{d}, \overline{d}], e = [\underline{e}, \overline{e}], f = [\underline{f}, \overline{f}],$$

and for simplicity, assume

$$\underline{a}, \overline{a} \geq 0, \underline{b}, \overline{b} \geq 0, \underline{c}, \overline{c} \geq 0, \underline{d}, \overline{d} \geq 0, \underline{e}, \overline{e} \geq 0, \underline{f}, \overline{f} \geq 0, a \neq d.$$

2.2. Fuzzy arithmetic

Let us recall some basic principles of fuzzy arithmetic needed for our further analysis. As the direct outcome of the basic definition, the following expressions were obtained (Alefeld & Herzberger, 1983).

$$[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}],$$

$$[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}],$$

$$[x]*[y]=\left[\min\left\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\right\},\max\left\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\right\}\right]$$
$$\frac{[x]}{[y]}=[\underline{x},\overline{x}]*\left[\frac{1}{\overline{y}},\frac{1}{\underline{y}}\right],0\notin[y].$$

2.3. The idea of interval extended zero method

The technique of interval extended zero method for solving interval equations is based on the fuzzy extension principle (Chalco-Cano *et al.*, 2007) which has been developed in (Seastjanov & Dymova, 2007) and (Seastjanov & Dymova, 2009). The values of uncertain parameters in an equation are substituted for corresponding intervals or fuzzy values and all arithmetic operations are substituted for relevant interval fuzzy operations. In general, for an arbitrary form of membership function the technique of fuzzy-interval calculations is based on the representation of initial fuzzy intervals in form of so-called α-cut (Hanss, 2005), (Dutta *et al.*, 2011) which are the intervals associated with the corresponding degrees of membership. All calculations are made with those α-cut according to the well known interval arithmetic rules and the resulting fuzzy intervals are obtained as the set of the corresponding final α-cut.

Consider the non-interval algebraic equation

$$g(x)=0.$$

The interval extension of the equation is obtained by replacement of all variables and arithmetic operations with intervals and relevant interval operations, respectively. As a result, we can present an interval equation as $[g]([x])=0$. In this equation, the right part is non interval degenerated the zero value, whereas the left part has an interval value form. Clearly, if $[g]([x])=[\bar{g},\underline{g}]$, then the equation $[g]([x])=0$ is correct only if $\overline{g}=\underline{g}=0$. In conventional interval analysis, it is usually assumed any interval containing zero may be considered as "interval zero". Here, an operation of "interval zero extension" is proposed to obtain "interval zero" in the right side of the question. To aim this, let us look at the problem from another point of view. We can define the result of operation $[c]-[c]$ as an "interval zero", where $[c]$ is an interval. Then, we get

$$[\underline{c},\overline{c}]-[\underline{c},\overline{c}]=[\underline{c}-\overline{c},\overline{c}-\underline{c}].$$

So the result of interval subtraction $[c]-[c]$ is an interval centered around zero. As a result, what we can say for the equation $g(x)=0$ is that the right hand side should be a symmetric interval with respect to zero with not defined width. Therefore, we can assume zero as $[-y,y]$ which is an interval centered around zero. This is the reason why this scheme is called interval extended zero (Dymova, 2007).

3. Solution of the equation by interval extended zero method

Consider the following quadratic algebraic equation

$$ax^2+bx+c=dx^2+ex+f. \tag{1}$$

Assuming the uncertain parameters, Eq.1 can be rewritten to its interval form as

$$[\underline{a},\overline{a}][\underline{x}^2,\overline{x}^2]+[\underline{b},\overline{b}][\underline{x},\overline{x}]+[\underline{c},\overline{c}]=[\underline{d},\overline{d}][\underline{x}^2,\overline{x}^2]+[\underline{e},\overline{e}][\underline{x},\overline{x}]+[\underline{f},\overline{f}]. \tag{2}$$

Then by manipulating Eq.2, one may write

$$[\underline{ax}^2,\overline{ax}^2]+[\underline{bx},\overline{bx}]+[\overline{c},\underline{c}]+[-\overline{dx}^2,-\underline{dx}^2]+[-\overline{ex},\underline{ex}]+[-\overline{f},-\underline{f}]=[-y,y], \tag{3}$$

and hence the following two equations are obtained

$$\underline{ax}^2+\underline{bx}+\underline{c}-\overline{dx}^2-\overline{ex}-\overline{f}=-y,$$
$$\overline{ax}^2+\overline{bx}+\overline{c}-\underline{dx}^2-\underline{ex}-\underline{f}=y. \tag{4}$$

Summing up both sides of Eq.4 implies

$$\underline{ax}^2+\underline{bx}+\underline{c}-\overline{d}\overline{x^2}-\overline{ex}-\overline{f}+\overline{ax}^2+\overline{bx}+\overline{c}-\underline{dx}^2-\underline{ex}-\underline{f}=0. \tag{5}$$

To solve Eq.5, firstly we assume that $\underline{x}=\overline{x}=x_m$, therefore

$$(\underline{a}-\overline{d}+\overline{a}-\underline{d})x_m^2+(\underline{b}-\overline{e}+\overline{b}-\underline{e})x_m+(\underline{c}-\overline{f}+\overline{c}-\underline{f})=0. \tag{6}$$

Thanks to the classical method for solving an algebraic quadratic equation form, the data parameter is computed as

$$\Delta=(\underline{b}-\overline{e}+\overline{b}-\underline{e})^2-4(\underline{a}-\overline{d}+\overline{a}-\underline{d})(\underline{c}-\overline{f}+\overline{c}-\underline{f}). \tag{7}$$

The Eq.6 is a quadratic form, therefore to find the solution, the following three cases, according to value of Δ parameter, are considered.

Case 1: $\Delta > 0$.

The Eq.6 has two roots as following

$$x_{m1}, x_{m2} = \frac{-(\underline{b} - \overline{e} + \overline{b} - \underline{e}) \pm \sqrt{\Delta}}{2(\underline{a} - \overline{d} + \overline{a} - \underline{d})}. \qquad (8)$$

Without loss of generality, let $x_{m1} < x_{m2}$, therefore x_{m1} is the lower and upper bound for $\overline{x}_{m1}$ and $\underline{x}_{m1}$, respectively. Similarly, x_{m2} is the lower and upper bound for $\underline{x}_{m2}$ and $\overline{x}_{m2}$, respectively. Next, we try to find the lower bounds for $\underline{x}_{m1}$ and $\underline{x}_{m2}$ and the upper bounds for $\overline{x}_{m1}$ and $\overline{x}_{m2}$.

Applying the fuzzy arithmetic for Eq.2, the following equation is obtained.

$$[\underline{a} - \overline{d}, \overline{a} - \underline{d}][\underline{x}^2, \overline{x}^2] + [\underline{b} - \overline{e}, \overline{b} - \underline{e}][\underline{x}, \overline{x}] + [\underline{c} - \overline{f}, \overline{c} - \underline{f}] = 0. \qquad (9)$$

Therefore, Eq.9 is rewritten as

$$\begin{aligned} (\underline{a} - \overline{d})\underline{x}^2 + (\underline{b} - \overline{e})\underline{x} + (\underline{c} - \overline{f}) &= 0, \\ (\overline{a} - \underline{d})\overline{x}^2 + (\overline{b} - \underline{e})\overline{x} + (\overline{c} - \underline{f}) &= 0. \end{aligned} \qquad (10)$$

The delta parameters are computed for both equations of Eq.10 as

$$\begin{aligned} \Delta\overline{x} &= (\underline{b} - \overline{e})^2 - 4(\underline{a} - \overline{d})(\underline{c} - \overline{f}), \\ \Delta\underline{x} &= (\overline{b} - \underline{e})^2 - 4(\overline{a} - \underline{d})(\overline{c} - \underline{f}). \end{aligned} \qquad (11)$$

Based on the obtained values for $\Delta\overline{x}$ and $\Delta\underline{x}$, the following options are considered.

(I) $\Delta\underline{x} > 0$ and $\Delta\overline{x} > 0$.

In this case, two equations in (10) have two answers like $\underline{x}_1, \underline{x}_2$ and $\overline{x}_1, \overline{x}_2$. Thus, we have

$$\begin{aligned} [\underline{x}_{m1}] &= [\underline{x}_1, x_{m1}], \qquad [\overline{x}_{m1}] = [x_{m1}, \overline{x}_1], \\ [\underline{x}_{m2}] &= [\underline{x}_2, x_{m2}], \qquad [\overline{x}_{m2}] = [x_{m2}, \overline{x}_2], \end{aligned} \qquad (12)$$

where $\underline{x}_1 < \underline{x}_2$ and $\overline{x}_1 < \overline{x}_2$.

Notice that if $\overline{x}_1 < x_{m1}$ or $\underline{x}_2 > x_{m2}$, set

$$[\overline{x}_{m1}] = \left[x_{m1}, \frac{x_{m1} + x_{m2}}{2}\right], \qquad [\underline{x}_{m2}] = \left[\frac{x_{m1} + x_{m2}}{2}, x_{m2}\right]. \qquad (13)$$

(II) $\Delta\underline{x} = 0$ and $\Delta\overline{x} = 0$

Let the roots of two equations in (10) be as $\underline{x}_1 = \underline{x}_2 = \underline{x}_{12}$ and $\overline{x}_1 = \overline{x}_2 = \overline{x}_{12}$. If $\underline{x}_{12} < x_{m1}$ then

$$[\underline{x}_{m1}] = [\underline{x}_{12}, x_{m1}], \qquad [\underline{x}_{m2}] = \left[\frac{x_{m1} + x_{m2}}{2}, x_{m2}\right]. \qquad (14)$$

On the other hand if $x_{m1} < \underline{x}_{12} < x_{m2}$ then

$$[\underline{x}_{m1}] = [x_{m1}, x_{m1}], \qquad [\underline{x}_{m2}] = [\underline{x}_{12}, x_{m2}]. \qquad (15)$$

If $\underline{x}_{12} > x_{m2}$, let

$$[\underline{x}_{m1}] = [x_{m1}, x_{m1}], \qquad [\underline{x}_{m2}] = \left[\frac{x_{m1} + x_{m2}}{2}, x_{m2}\right]. \qquad (16)$$

For the case $\overline{x}_{12} > x_{m2}$, then

$$[\overline{x}_{m1}] = \left[x_{m1}, \frac{x_{m1} + x_{m2}}{2}\right], \qquad [\overline{x}_{m2}] = [x_{m2}, \overline{x}_{12}]. \qquad (17)$$

If $x_{m1} < \overline{x}_{12} < x_{m2}$, let

$$[\underline{x}_{m1}] = [x_{m1}, x_{m1}], \qquad [\overline{x}_{m1}] = [x_{m1}, \overline{x}_{12}], \qquad [\overline{x}_{m2}] = [x_{m2}, x_{m2}], \qquad (18)$$

and if $\overline{x}_{12} < x_{m1}$, then the interval solutions are

$$[\overline{x}_{m1}] = \left[x_{m1}, \frac{x_{m1} + x_{m2}}{2}\right], \qquad [\overline{x}_{m2}] = [x_{m2}, x_{m2}]. \qquad (19)$$

(III) $\Delta\underline{x} < 0$ and $\Delta\overline{x} < 0$.
The two equations in (10) have no real roots, so as a result, set

$$[\underline{x}_{m1}] = [x_{m1}, x_{m1}],\ \ [\overline{x}_{m1}] = \left[x_{m1}, \frac{x_{m1} + x_{m2}}{2}\right], \tag{20}$$

$$[\underline{x}_{m2}] = \left[\frac{x_{m1} + x_{m2}}{2}, x_{m21}\right], \qquad [\overline{x}_{m2}] = [x_{m2}, x_{m2}].$$

The other cases for $\Delta\underline{x}$ and $\Delta\overline{x}$ can be treated in similar ways.
Now, let the interval solutions are

$$[\underline{x}_{m1}] = \left[\underline{x}'_1, x_{m1}\right], \qquad [\overline{x}_{m1}] = [x_{m1}, \overline{x}'_1], \tag{21}$$

$$[\underline{x}_{m2}] = \left[\underline{x}'_2, x_{m2}\right], \qquad [\overline{x}_{m2}] = [x_{m2}, \overline{x}'_2].$$

Obviously, if $\underline{x}_{m1}$ and $\underline{x}_{m2}$ be minimal in the intervals $[\underline{x}_{m1}] = [\underline{x}'_1, x_{m1}]$ and $[\underline{x}_{m2}] = \left[\underline{x}'_2, x_{m2}\right]$, respectively, the maximal values of $\overline{x}_{m1}$ and $\overline{x}_{m2}$ are obtained by substituting $\underline{x}_{m1}$ and $\underline{x}_{m2}$ in Eq.5.
Substituting $\underline{x}'_1$ and $\underline{x}'_2$ in Eq.5, the following equations are achieved.

$$\underline{ax}'^2_1 + \underline{bx}'^2_1 + \underline{c} - \overline{dx}^2 - \overline{ex} - \overline{f} + \overline{ax}^2 + \overline{bx} + \overline{c} - \underline{dx}'^2_1 - \underline{ex}'_1 - \underline{f} = 0, \tag{22}$$

$$\underline{ax}'^2_2 + \underline{bx}'^2_2 + \underline{c} - \overline{dx}^2 - \overline{ex} - \overline{f} + \overline{ax}^2 + \overline{bx} + \overline{c} - \underline{dx}'^2_2 - \underline{ex}'_2 - \underline{f} = 0. \tag{23}$$

Suppose that $\overline{x}'_{m1}$ and $\overline{x}''_{m1}$ are the solutions of quadratic equation (22) for $\Delta'_1 > 0$ and the solutions of equation (23) be $\overline{x}'_{m2}, \overline{x}''_{m2}$ for $\Delta'_2 > 0$. Therefore, $\overline{x}^U_{m1.max}$ and $\overline{x}^U_{m2.max}$ are defined as follows

$$\overline{x}^U_{m1.max} = \max\{\overline{x}'_1, \overline{x}'_{m1}, \overline{x}''_{m1}\}, \tag{24}$$

$$\overline{x}^U_{m2.max} = \max\{\overline{x}'_2, \overline{x}'_{m2}, \overline{x}''_{m2}\}.$$

If $\Delta'_1 = \Delta'_2 = 0$ then $\overline{x}'_{m1} = \overline{x}''_{m1}$ and $\overline{x}'_{m2} = \overline{x}''_{m2}$, so

$$\overline{x}^U_{m1.max} = \max\{\overline{x}'_1, \overline{x}'_{m1}\}, \tag{25}$$

$$\overline{x}^U_{m2.max} = \max\{\overline{x}'_2, \overline{x}'_{m2}\}.$$

It is easy to obtain

$$\overline{x}^U_{m1.max} = \overline{x}'_1 \ , \quad \overline{x}^U_{m2.max} = \overline{x}'_2, \tag{26}$$

for the case $\Delta'_1 < \Delta'_2 < 0$.
Similarly, we get the minimum of $\underline{x}_{m1}$ and $\underline{x}_{m2}$ by substituting of the maximal values of $\overline{x}_{m1}$ and $\overline{x}_{m2}$ in equation (5) as

$$\underline{ax}^2 + \underline{bx} + \underline{c} - \overline{dx}'^2_1 - \overline{ex}'_1 - \overline{f} + \overline{ax}'^2_1 + \overline{bx}'_1 + \overline{c} - \underline{dx}^2 - \underline{ex} - \underline{f} = 0, \tag{27}$$

$$\underline{ax}^2 + \underline{bx} + \underline{c} - \overline{dx}'^2_2 - \overline{ex}'_2 - \overline{f} + \overline{ax}'^2_2 + \overline{bx}'_2 + \overline{c} - \underline{dx}^2 - \underline{ex} - \underline{f} = 0. \tag{28}$$

Suppose that for $\Delta''_1 > 0$ and $\Delta''_2 > 0$, the solutions for Eq.27 and Eq.28 are $\underline{x}'_{m1}, \underline{x}''_{m1}$ and $\underline{x}'_{m2}$, $\underline{x}''_{m2}$, respectively. So, $\underline{x}^L_{m1.min}$ and $\underline{x}^L_{m2.min}$ are defined as follows:

$$\underline{x}^L_{m1.min} = \min\{\underline{x}'_1, \underline{x}'_{m1}, \underline{x}''_{m1}\}, \tag{29}$$

$$\underline{x}^L_{m2.min} = \min\{\underline{x}'_2, \underline{x}'_{m2}, \underline{x}''_{m2}\}.$$

If $\Delta''_1 = \Delta''_2 = 0$ then $\underline{x}'_{m1} = \underline{x}''_{m1}$ and $\underline{x}'_{m2} = \underline{x}''_{m2}$ therefore, we have

$$\underline{x}^L_{m1.min} = \min\{\underline{x}'_1, \underline{x}'_{m1}\}, \tag{30}$$

$$\underline{x}^L_{m2.min} = \min\{\underline{x}'_2, \underline{x}'_{m2}\}.$$

Considering $\Delta''_1 < 0,\ \Delta''_2 < 0$, we get

$$\underline{x}^L_{m1.min} = \underline{x}'_1 \ , \quad \underline{x}^L_{m2.min} = \underline{x}'_2. \tag{31}$$

Finally, the following interval solutions are obtained.

$$[\underline{x}_{m1}] = [\underline{x}^L_{m1.min}, x_{m1}], \qquad [\overline{x}_{m1}] = [x_{m1}, \overline{x}^U_{m1.max}], \tag{32}$$
$$[\underline{x}_{m2}] = [\underline{x}^L_{m2.min}, x_{m2}], \qquad [\overline{x}_{m2}] = [x_{m2}, \overline{x}^U_{m2.max}].$$

Case 2: $\Delta = 0$.
The Eq.6 has the following root

$$x_m = x_{m1} = x_{m2} = \frac{-(\underline{b} - \overline{e} + \overline{b} - \underline{e})}{2(\underline{a} - \overline{d} + \overline{a} - \underline{d})}, \tag{33}$$

Hence, x_m is the upper bound for $\underline{x}_m$ and lower bound for $\overline{x}_m$.
Referring to Eq.11, the delta parameters are calculated for both quadratic equations in Eq.10. Similarly, there are the following three cases.
(I) $\Delta\underline{x} > 0$ and $\Delta\overline{x} > 0$.
Let the roots of two equations in Eq.10 are denoted by $\underline{x}_1, \underline{x}_2$ and $\overline{x}_1, \overline{x}_2$, respectively.
If $\underline{x}_1 < \underline{x}_2 < x_m$ and $x_m < \overline{x}_2 < \overline{x}_1$ then

$$[\underline{x}_m] = [\overline{x}_2, x_m], \qquad [\overline{x}_m] = [x_m, \overline{x}_2], \tag{34}$$

and if $\underline{x}_1 < x_m < \underline{x}_2$ and $\overline{x}_2 < x_m < \overline{x}_1$ then

$$[\underline{x}_m] = [\underline{x}_1, x_m], \qquad [\overline{x}_m] = [x_m, \overline{x}_1] \tag{35}$$

Also, if $x_m < \underline{x}_1 < \underline{x}_2$ and $\overline{x}_2 < \overline{x}_1 < x_m$ then

$$[\underline{x}_m] = [x_m, x_m], \qquad [\overline{x}_m] = [x_m, x_m]. \tag{36}$$

Similarly, the other cases easily can be constructed.

(II) $\Delta\underline{x} = 0$ and $\Delta\overline{x} = 0$.
Let the roots of two quadratic equations in Eq.10 are $\underline{x}_{12} = \underline{x}_1 = \underline{x}_2$ and $\overline{x}_1 = \overline{x}_2 = \overline{x}_{12}$, respectively. Therefore, if $\underline{x}_{12} < x_m < \overline{x}_{12}$ then

$$[\underline{x}_m] = [\underline{x}_{12}, x_m], \qquad [\overline{x}_m] = [x_m, \overline{x}_{12}], \tag{37}$$

and if $x_m < \underline{x}_{12}$ and $x_m < \overline{x}_{12}$, then

$$[\underline{x}_m] = [x_m, x_m], \qquad [\overline{x}_m] = [x_m, \overline{x}_{12}]. \tag{38}$$

Also, if $x_m < \underline{x}_{12}$ and $\overline{x}_{12} < x_m$, then we have

$$[\underline{x}_m] = [\overline{x}_m] = [x_m, x_m]. \tag{39}$$

(III) $\Delta\underline{x} < 0$ and $\Delta\overline{x} < 0$.
The intervals solutions are $[\underline{x}_m] = [\overline{x}_m] = [x_m, x_m]$.
Other cases for $\Delta\underline{x}, \Delta\overline{x}$ are treated similarly.
Now, let the interval solutions are $[\underline{x}_m] = [\underline{x}_1', x_m]$ and $[\overline{x}_m] = [x_m, \overline{x}_1']$. Substituting $\underline{x}_1'^2$ and $\underline{x}_2'^2$ in Eq.5, the two following equations are obtained.

$$\underline{a}\underline{x}_1'^2 + \underline{b}\underline{x}_1'^2 + \underline{c} - \overline{d}\overline{x}^2 - \overline{e}\overline{x} - \overline{f} + \overline{a}\overline{x}^2 + \overline{b}\overline{x} + \overline{c} - \underline{d}\underline{x}_1'^2 - \underline{e}\underline{x}_1' - \underline{f} = 0, \tag{40}$$
$$\underline{a}\underline{x}_2'^2 + \underline{b}\underline{x}_2'^2 + \underline{c} - \overline{d}\overline{x}^2 - \overline{e}\overline{x} - \overline{f} + \overline{a}\overline{x}^2 + \overline{b}\overline{x} + \overline{c} - \underline{d}\underline{x}_2'^2 - \underline{e}\underline{x}_2' - \underline{f} = 0. \tag{41}$$

If $\Delta'_1 > 0$ and $\Delta'_2 > 0$, let $\overline{x}_m'$ and $\overline{x}_m''$ the roots of Eq.40 and $\underline{x}_m', \underline{x}_m''$ the roots of Eq.41. Thus,

$$\overline{x}^U_{m.max} = \max\{\overline{x}_1', \overline{x}_m', \overline{x}_m''\}, \tag{42}$$
$$\underline{x}^L_{m.min} = \min\{\underline{x}_1', \underline{x}_m', \underline{x}_m''\}.$$

If $\Delta'_1 = \Delta'_2 = 0$ then $\overline{x}_m' = \overline{x}_m''$ and $\underline{x}_m' = \underline{x}_m''$ therefore,

$$\overline{x}^U_{m.max} = \max\{\overline{x}_1', \overline{x}_m' = \overline{x}_m''\}, \tag{43}$$
$$\underline{x}^L_{m.min} = \min\{\underline{x}_1', \underline{x}_m' = \underline{x}_m''\}.$$

At last if $\Delta'_1 < 0$ and $\Delta'_2 < 0$, then

$$\overline{x}^U_{m.max} = \overline{x}_1', \qquad \underline{x}^L_{m.min} = \underline{x}_1'. \tag{44}$$

Finally, we get the following interval solutions as

$$[\underline{x}_m] = [\underline{x}^L_{m.min}, x_m], \qquad [\overline{x}_m] = [x_m, \overline{x}^U_{m.max}]. \qquad (45)$$

Case 3: $\Delta < 0$.
There are no interval solutions.

4. Numerical Examples

In this section, three examples with different uncertain parameters are provided to illustrate the simplicity and applicability of the presented method.

Example 1. Consider the following quadratic algebraic equation with interval parameters

$$ax^2 + bx + c = dx^2 + ex + f,$$

where, $a = [3,4], b = [1,2], c = [3,4], d = [1,2], e = [6,7], f = [1,2]$.
To solve this problem, consider

$$(\underline{a} - \overline{d} + \overline{a} - \underline{d})x_m^2 + (\underline{b} - \overline{e} + \overline{b} - \underline{e})x_m + \left(\underline{c} - \overline{f} + \overline{c} - \underline{f}\right) = 0.$$

According to the assumptions, we get

$$4x_m^2 - 10x_m + 4 = 0.$$

Thus, the solutions are

$$x_{m1} = \frac{1}{2}, \quad x_{m2} = 2.$$

The delta parameters for both equations of Eq.10 are calculated as

$$\Delta\underline{x} = 32, \qquad \Delta\overline{x} = -20 < 0,$$

and

$$\underline{x}_1 = 0.2 , \quad \underline{x}_2 = 5.8 .$$

So, we get

$$[\underline{x}_{m1}] = [0.2,0.5], \qquad [\overline{x}_{m1}] = [0.5,0.5],$$
$$[\underline{x}_{m2}] = [1.25,2], \qquad [\overline{x}_{m2}] = [2,2].$$

Substituting $\underline{x}'_1 = 0.2$ and $\underline{x}'_2 = 1.25$ in Eq.22 and Eq.23, respectively, the following results are obtained.

$$\Delta'_1 = 0.36, \qquad \overline{x}'_{m1}, \overline{x}''_{m1} = 1.1,1.4, \qquad \overline{x}^U_{m1.max} = \max\{0.5,1.1,1.4\} = 1.4,$$
$$\Delta'_2 = 18, \qquad \overline{x}'_{m2}, \overline{x}''_{m2} = 0.2,2.3, \qquad \overline{x}^U_{m2.max} = \max\{2,0.2,2.3\} = 2.3,$$

and by substituting the lower bounds of $\overline{x}_{m1}$ and $\overline{x}_{m2}$, that is, 0.5 and 2 in Eq.27 and Eq.28, respectively, we achieve

$$\Delta''_1 = 9, \quad \underline{x}'_{m1} = 0.5 , \underline{x}''_{m1} = 2 \ , \underline{x}^L_{m1.min} = \min\{0.2,2,0.5\} = 0.2 ,$$
$$\Delta''_2 = 9, \quad \underline{x}'_{m2} = 0.5 , \underline{x}''_{m2} = 2 \ , \underline{x}^L_{m2.min} = \min\{1.25,2,0.5\} = 0.5.$$

Therefore, the interval solution are as follows.

$$[\underline{x}_{m1}] = [0.2,0.5], \qquad [\overline{x}_{m1}] = [0.5,1.4],$$
$$[\underline{x}_{m2}] = [0.5,2], \qquad [\overline{x}_{m2}] = [2,2.3].$$

Example 2. Let $a = [0.1,3], b = [1,9], c = [3,8], d = [2,5], e = [1,5], f = [1,7]$, for the following quadratic algebraic equation with interval parameters

$$ax^2 + bx + c = dx^2 + ex + f.$$

Considering the quadratic algebraic equation

$$-3.9x_m^2 + 4x_m + 3 = 0.$$

Delta parameter and the roots are calculated as

$$\Delta = 62.8 , \qquad x_{m1} = -0.50316 , \qquad x_{m2} = 1.528801,$$

and

$$\Delta\underline{x} = -62.4 < 0 \ , \quad \Delta\overline{x} = 36 > 0 \ , \quad \overline{x}_1 = -7 \ , \quad \overline{x}_2 = -1.$$

Furthermore

$$[\underline{x}_{m1}] = [-0.50316,0.50316], \qquad [\overline{x}_{m1}] = [-0.50316, -0.50316],$$
$$[\underline{x}_{m2}] = [1.528801,1.528801], \qquad [\overline{x}_{m2}] = [1.528801,1.528801].$$

In the other side,

$$\Delta'_1 = 36.151816\,, \quad \overline{x}'_{m1} = -0.50316\,, \quad \overline{x}''_{m1} = 2.50316.$$

So,

$$\overline{x}^U_{m1.max} = \max\{-0.50316, 2.50316, -0.50316\} = 2.50316\,,$$

and

$$\Delta''_1 = 4.474064\;, \quad \overline{x}'_{m2} = 0.4712\;, \quad \overline{x}''_{m2} = 1.528799,$$

$$\overline{x}^U_{m2.max} = \max\{1.528801, 1.528799.0.4712\} = 1.528801.$$

Calculating $\overline{x}'_{m1}$, $\overline{x}''_{m1}$, $\overline{x}'_{m2}$ and $\overline{x}''_{m2}$, we find

$$\underline{x}^L_{m1.min} = \min\{-0.50316, 0.50316, -0.50316\} = -0.50316\;,$$

$$\underline{x}^L_{m2.min} = \min\{1.528801, 1.528801, -1.528801\} = -1.528801.$$

Therefore,

$$[\underline{x}_{m1}] = [-0.50316, 0.50316], \qquad [\overline{x}_{m1}] = [-0.50316, -2.50316],$$

$$[\underline{x}_{m2}] = [-1.528801, 1.528801], \qquad [\overline{x}_{m2}] = [1.528801, 1.528801].$$

Example 3. Let $a = [3,4], b = [1,2], c = [1,2], d = [1,2], e = [4,5], f = [3,4]$. Find the interval solutions for the following quadratic algebraic equation

$$ax^2 + bx + c = dx^2 + ex + f.$$

The quadratic algebraic equation related to the given values are achieved as

$$4x_m^2 - 6x_m - 4 = 0.$$

Considering the process in section 3, the following results are obtained.

$$x_{m1} = -0.5\,, \quad x_{m2} = 2\,, \quad \Delta\underline{x} = 28\,, \quad \Delta\overline{x} = 16,$$

$$\underline{x}_1 = -0.64\,, \quad \underline{x}_2 = 4.64\,, \quad \overline{x}_1 = 0\;, \quad \overline{x}_2 = \frac{4}{3},$$

and

$$[\underline{x}_{m1}] = [-0.64, -0.5]\,, \quad [\overline{x}_{m1}] = [-0.5, 0],$$

$$[\underline{x}_{m2}] = [0.75, 2]\,, \quad [\overline{x}_{m2}] = [2, 2].$$

Following the mentioned process, we get

$$\overline{x}^U_{m1.max} = \overline{x}_1 = 0\;, \quad \overline{x}^U_{m2.max} = \max\{2, -1, 2.5\} = 2.5.$$

Similarly

$$\underline{x}^L_{m1.min} = \min\{-0.64, -0.85, 2.35\} = -0.85,$$

$$\underline{x}^L_{m2.min} = \min\{0.75, -0.5, 2\} = -0.5.$$

Therefore, the interval solutions are obtained as follows.

$$[\underline{x}_{m1}] = [-0.85, -0.5]\,, \quad [\overline{x}_{m1}] = [-0.5, 0],$$

$$[\underline{x}_{m2}] = [0.5, 2]\,, \quad [\overline{x}_{m2}] = [2, 2.5].$$

5. Conclusion

The objective of the paper is to present a new concept to the solution of interval and fuzzy quadratic equations of dual form based upon the generalized technique of interval and fuzzy extension called "interval extended zero" method. The key idea is the dealing of "interval zero" as a symmetrical interval around zero. It is presented that the idea of such approach is a direct consequence of interval subtraction operation. The method makes it easy to solve some formerly unresolved methodological problems in applied interval analysis and fuzzy arithmetic. An important for practice advantage of the idea of this method is that it provides substantially narrower solutions than conventional methods. The effectiveness and applicability of the derived interval method was demonstrated by three numerical examples.

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