Research Article

A Comprehensive Analysis of the Impact of System Parameters on Subspace-based DoA Estimation Performance

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Abstract
This work provides an explanatory analysis of the influence of input parameters on the performance of subspace-based Direction of Arrival (DoA) estimation algorithms. The objective of this work is twofold. First, to drive a Steering Vector (SV) that works for arbitrary array configuration rather than just Uniform Linear Array (ULA) geometry. Second, to identify how the performance of the subspace-based algorithms is affected by tuning the input parameters. The later objective is crucial as it allows optimizing the algorithm through selecting optimum parameters to set an appropriate tradeoff between complexity and performance based on the intended applications. Toward that end, we firstly drive an SV for arbitrary array configuration followed by revealing the working principle of subspace based DoA techniques. Secondly, we evaluate the impact of several parameters namely Signal to Noise Ratio (SNR), number of snapshots, number of array elements, separation between array elements, number of available sources, and dependency between sources to conduct our analysis. Numerical simulations over a wide range of scenarios along with intensive Monte Carlo simulations are conducted to show the influence of these parameters on the resolution, accuracy, and complexity of the subspace based DoA estimation algorithm. As demonstrated by the obtained results, the performance of this class of DoA estimation method is determined mostly by the values of the input parameters. Furthermore, the simulation results show that tradeoff between performance and computational complexity needs to be considered when the system parameters are chosen for DoA estimation algorithms.

Keywords: DoA, Subspace, MUSIC, Estimation, Computational-Complexity, Resolution.

1. Introduction
Estimating the signals' parameters, such as polarization, frequency, and Direction of Arrival (DoA), is a fundamental problem in array signal processing and has attracted the research-communities' attention particularly in far-field signal applications. DoA estimation refers to the estimating the direction in which the signals incident on the antenna array, and it plays an essential role in localization-based services and several engineering applications, including public security (Wan et al., 2016), emergency call localization (Kabir et al., 2009), sonar (Bardhan & Jacob, 2015), radar systems (Kim & Lee, 2018), bearing tracking systems (Guo et al., 2016) , and wireless communications (Arceo-Olague et al., 2010). Furthermore, the performance of the newly developed smart antenna systems mainly relies on the accuracy of the DoA

measurement such that the DoA information obtained from the estimation algorithm is passed to the adaptive beamforming technique to maximize the gain towards the desired user(s) and suppressing the beam (i.e., producing nulls) in the direction of interferer(s) (Ghali & Abdullah, 2011).

DoA estimation has been an attractive-research topic since the introduction of radar systems in last century and it has a rich-history starting from mechanically-rotated antennas to indicate the direction of emitted signals (Krim & Viberg, 1996). Several algorithms and methodologies exist in literature to extract DoA information from the collected signals at the output of the antenna arrays.

One of the earliest DoA estimation algorithms is the Minimum Variance Distortion-less Response (MVDR) proposed by Capon in (Capon, 1969). This method measures the strength of the signals in all directions and the direction from which the gain of the beamformer equals to one is selected as the direction of the signal. Although, the MVDR has a high resolution, it is computationally inefficient as it requires to take the inverse of covariance matrix (CM) and its performance degrades when incident signals are correlated and/or the number of data snapshots is limited.

The earliest approach that explores the eigenvectors of the measured CM to obtain DoA information of the arrived signals is the Pisarenko Harmonic Decomposition (PHD) (Pisarenko, 1973). This method assumes that one column is the true dimension of the noise without considering the actual number of emitted sources (Wax et al.,1984). However, the PHD method has two main limitations. First, it is applicable to Uniform Linear Array (ULA) only. Second, it has a high percentage of false detections.

Later, to overcome the above problems, a new class of DoA estimation techniques known as subspace-based algorithm was emerged. Unlike previous methods which used the statistical properties of the received signals, subspace-based methods investigate the eigen structure of the collected data to estimate incident angle accurately.

Minimum Norm (MinNorm) algorithm was proposed by Reddi (1979) and later enhanced by the authors in (Kumaresan & Tufts, 1983) which is an earliest version of the subspace-based algorithms. MinNorm technique tries to minimize the array output norm via the best array weights vector. This algorithm, however, is applicable only to ULA configuration.

The most commonly used and supper-resolution technique within this class is the Multiple Signal Classification (MUSIC) algorithm (Schmidt, 1986). To estimate DoAs of the incoming signals using this algorithm, it is required to factorize the CM into two subspaces, namely Noise SubSpace (NSS) and Signal SubSpace (SSS). Then, the orthogonality between the array Steering Vector (SV) and the NSS is explored to construct the spatial spectrum of the arrived signals. The peaks of the spectrum are the indications for the DoAs of the signals. Although MUSIC has a super-resolution feature, high computational complexity to construct subspaces is the main drawback of this algorithm.

To address this issue, computationally efficient root MUSIC algorithm was proposed (Barabell, 1983; Liu et al., 2018). In this method, intensive searching through the array SV was replaced by a low cost searching through the roots of polynomials. However, this methodology can be applied to the ULA geometry only.

Besides its high complexity, traditional MUSIC algorithm is more sensitive to correlated signals. To tackle this problem, Spatial Smoothing (SS) technique was introduced (Evans et al., 1982). The SS technique was further investigated by Pillai and Kwon (1989) for correlated signal identifications. The principle of this technique is to divide the ULA into multiple overlapped subarrays in order to introduce phase shifts between arrived signals. Subsequent studies in (Qian et al., 2014) applied such methodologies to remove the dependency among the received signals in order to improve the performance of MUSIC algorithm for correlated signals. Recently, Enhanced SS (ESS) technique for DoA estimation of coherent signals was proposed by (Pan et al., 2020). It is claimed that the existing ESS techniques by (Du & Kirlin, 1991; Dong et al., 2007) do not take full benefit from information of the subarrays. Therefore, the new ESS is directly applied to the SSS (ESS-SSS) which contains full information regarding the directions of the incident signals. Compared to the previous smoothing techniques, ESS-SSS retains two main features. First, it explores all the information within the SSS. Second, it has higher noise immunity. These, of course, result more accurate estimations. However, combining the SS technique with MUSIC algorithm adds a higher complexity to the algorithms' procedure.

A new bearing estimation algorithm known as Projection Matrix (PM) was proposed in (Yeh, 1986) . Although PM is classified as a subspace algorithm, it does not require factorization of CM. Alternatively, this method explores L columns (where L is the number of sources) of the CM to construct the PM. The idea of this technique stems from the fact the PM construction requires less computation compared to the eigenvector-based techniques. However, selecting different columns withing the CM alters the estimation resolution, and no optimum criterion has been suggested to choose the column positions.

A high-resolution algorithm to localize incoherent, partially coherent, and coherent narrowband sources was presented in (Cadzow, 1988). This algorithm applies an iterative technique to solve a non-liner relationship and determines the angle of the correlated signals. It is shown that, such iterative technique allows performance improvement in comparison to the MUSIC algorithm with SS technique for correlated signals.

Another subspace DoA method is the Estimation of Signal Parameters via Rotational Invariance (ESPRIT) technique (Roy & Kailath, 1989). So that ESPRIT works, the number of sources must not exceed the number of antenna elements in the array. Additionally, it is infeasible with large array size due to its high computational complexity resulted from applying eigenvalue decomposition (EVD) technique.

To avoid the complexity associated with matrix decomposition required for subspaces acquisition, propagator algorithm was introduced (Munier & Delisle, 1991). This method extracts the DoA information directly from the arrived signals without generating and decomposing the CM. The principle of this algorithm is derived by the fact that the complexity of CM generation is higher than that for propagator vector. The performance of this algorithm, however, is limited when the noise source is not Adaptive White Gaussian Noise (AWGN).

Thereafter, Orthogonal Propagator (OP) algorithm (Marcos et al., 1995) and a new DoA estimation technique in (Tayem, 2005) were proposed that show better performance compared to the traditional propagator method at the cost of high computational complexity.

A low complexity Root-MUSIC algorithm was proposed by integrating the idea of MinNorm with the Root-MUSIC algorithm (Ren & Willis, 1997). However, in this algorithm some of the poles mislead the estimator by pointing to the wrong directions particularly at low SNR.

An improved MUSIC-based DoA estimation method under multipath fading environment was proposed in (Jami & Ormondroyd, 2000). To eliminate the biased result in such environment, this improved scheme takes the impact of angular spread resulted from signal scattering into account in the computation of the SV. This new and scaled SV leads to better estimation when angular spread exists compared to the traditional methods.

The principle of subspace technique and spatial filtering were combined for source localization in (MacInnes, 2004). It is shown that such combination allows accurate DoA estimation in medium SNR scenarios and separate closely spaced sources which are not resolvable by the classical and adaptive beamforming techniques. Besides, the use of spatial filtering, which spatially filters the incoming snapshots, allows the algorithm to work well in overdetermined cases where the number of sources is greater than the array elements.

A low complexity version of propagator algorithm was proposed in (Chen et al., 2011). Unlike the standard propagator method, this technique uses partial CM to construct the propagator operator. Due to using the off-diagonal part of the CM, this algorithm also works well under nonuniform colored noise condition. It is shown that this modified algorithm achieves the same performance as MUSIC and original propagator algorithms at high SNR with significantly reduced complexity.

In Zhang et al. (2017), the authors enhanced the root-MUSIC algorithm that was applied to non-ULA configuration to estimate not only elevation but also azimuth angle.

To avoid the need of high-number of arithmetic operations in the standard MUSIC algorithm, real-valued MUSIC technique was proposed by Yan et al. (2018). The principle of this algorithm stems from the fact that multiplication between complex numbers is more expensive compared to real numbers. Thus, they could reduce the complexity of MUSIC algorithm by about 75%.

A novel DoA estimation technique was proposed based on a Polarization Sensitive Array (PSA) for an environment where coherent and non-coherent signals co-exist (Dong et al., 2021a). This method first estimates the DoA of the independent signal using quaternion root-MUSIC (Q-root-MUSIC) algorithm by employing the quaternion model. Then, a new CM containing the data for coherent signals only is formulated. Lastly, this new CM is processed by the root-MUSIC algorithm to resolve coherent signals. Compared to the conventional algorithms proposed for mixture (correlated and uncorrelated) signals, this method retains higher precision and lower complexity.

Recently, low-complexity MUSIC-based algorithm was proposed by Khan et al. (2021) by combining Instantons Frequency (IF) estimator with DoA estimation. This algorithm first estimates the IF of the signals impinging on the array. Then, the signal sources are separated by using the estimated IF and detect DoAs of the signals. Besides its low complexity, this technique works in overdetermined cases, where the number of sources is greater than the array elements.

Two improved versions of MUSIC algorithm, namely Total Array-based MUSIC (TA-MUSIC) and ESPRIT MUSIC (E-MUSIC) algorithms were proposed for Cubic Coprime Array (CCA) configuration (Gong & Chen, 2021). To maximize the Degree of Freedom (DoF), TA-MUSIC employs auto CM as well as mutual CM of the signals received from sub arrays. This led to higher estimation precision. However, the complexity of TA-MUSIC is higher than standard MUSIC algorithm due to computing two CMs. To address the high complexity of TA-MUSIC due to 2D spectral search, they combined ESPRIT with MUSIC algorithm and proposed E-MUSIC algorithm in which 2D spectral search has been transferred into 1D searching. From the obtained results, it is noticeable that TA-MUSIC has better performance than E-MUSIC. Therefore, a clear tradeoff between complexity and accuracy is observed.

Recently, Eranti & Barkana (2022) provided an overview regarding the impact of employing adaptive directional time-frequency distributions (ADTFD) on the DoA estimation performance. Particularly, they analyzed and compared the performance of MUSIC, ESPRIT, and EVD methods in two cases namely with and without employing ADTFD as a pre-processing step. It is found that the performance of MUSIC algorithm is significantly improved by combining it with ADTFD technique. Whereas this pre-processing technique does not show noticeable impact on ESPRIT and EVD methods. It is also observable that EVD algorithm outperformed MUSIC and ESPRIT methods when they are employed without ADTFD. It is worth mentioning that ADTFD raises computational complexity of the DoA estimation algorithms.

From the above detailed-description, standard MUSIC algorithm can be seen as a backbone for the majority of the recently developed DoA estimation techniques (Abdulrazak, 2018; Suksiri & Fukumoto, 2019; Eray & Temizel, 2020; Dong et al., 2021b; Liu et al., 2021; Abdelbari & Bilgehan, 2021; Gong & Chen, 2021; Eranti & Barkana, 2022) . Practically, various challenges face direction-finding systems such as variation of SNR caused by channel fading, array size constraints, limited number of available data samples known as snapshots, inter-element spacing between array elements, and correlation between sources. In order to achieve acceptable performance and set suitable tradeoff between resolution, accuracy, and computational complexity, it is essential to identify these challenges when direction-finding algorithms are implemented. Recent performance-analysis of the subspace-based methods can be found in (Boustani et al., 2019; Eray & Temizel, 2020; Abeida & Delmas, 2021) that each conducted the analysis from a specific point of view. To the best of the authors' knowledge, no such in-depth and general performance analysis can be found in literature that takes all the input variables into account.

Therefore, in this paper the concept of the most well-known subspace-based technique, MUSIC algorithm, is explored to analysis how estimation performance of this class of DoA method is impacted by the input parameters. The results, however, can be generalized to other subspace-based (MUSIC-like) techniques. Particularly, we take the impact of six parameters on the estimation performance into account, namely SNR variations, number of snapshots, number of array elements, spatial separation between array elements, number of available sources, and correlation between sources. To this end, numerical simulations over a wide range of scenarios along with intensive Monte Carlo simulations are performed to show the influence of these parameters on resolution, accuracy, and complexity of subspace-based DoA estimation techniques.

The main contributions of this work can be summarized as follows:

- A new SV is derived that is suitable not only for ULA, but also arbitrary array configuration.
- The impact of several parameters, used as input to the DoA estimation algorithms is critically evaluated on the performance of subspace-based algorithm. Additionally, the tradeoff between estimation performance and computational complexity is analyzed. This contribution is significantly important as it leads the researchers to optimize the algorithms' outputs and select optimum parameters to make an appropriate tradeoff between complexity and performance based on the intended applications. For example, if an application needs fast convergence without much constrain on complexity, one of possible solutions could be increasing the number of array elements to be input to the algorithm.

This paper is structured as follows. Section 2 presents the DoA problem formulation. While the principle of subspace-based DoA estimation is revealed in section 3. In section 4, the obtained results are presented and comprehensively discussed. Section 5 concludes the main points of the presented results.

2. DoA Problem Formulation

We adopt an arbitrary array consisting of M identical elements to collect incident signals transmitted by L far-field sources as illustrated in Figure (1). The arrived signals are assumed to be independent and narrowband, and they impinge on the array from random directions of $(\theta_l,\ \phi_l)_{\ l=1,2,\cdots L}$ where, θ_l and ϕ_l represent the elevation and azimuth angles of lth signal, respectively. To obtain DoA information, the collected signals at the output of the array are down-converted using Digital Down Convertor (DDC) and processed at the baseband-level. The data vector which composes of both desired signal and additive noise are represented as follows:

$$\boldsymbol{x}(t) = \boldsymbol{A}(\theta, \phi)\boldsymbol{s}(t) + \boldsymbol{n}(t)\ , \tag{1}$$

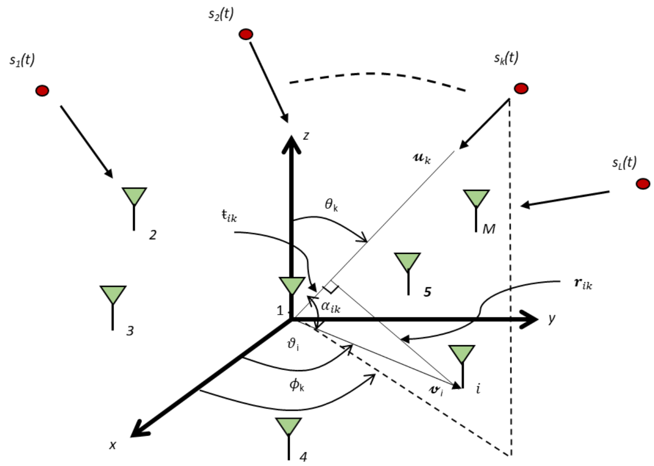


Figure 1. DoA problem formulation with M-element arbitrary array and L sources.

where, $\boldsymbol{s}(t) = [s_1(t) \;\; s_2(t) \;\; \cdots \;\; s_L(t)\,]^T \in \mathbb{C}^{L\,x\,1}$ donates modulated Binary Phase Shift Key (BPSK) signal, $\boldsymbol{n}(t) = [n_1(t),\; n_2(t) \;\; \cdots \; n_M(t)\,]^T \in \mathbb{C}^{M\,x\,1}$ represents the Additive White Gaussian Noise (AWGN) at each channel, and $\boldsymbol{A}(\theta,\phi)$ is a steering matrix which includes the SVs for L signals and defined as

$$\boldsymbol{A}(\theta,\phi) = [\boldsymbol{a}(\theta_1,\phi_1) \;\; \boldsymbol{a}(\theta_2,\phi_2) \;\; \cdots \;\; \boldsymbol{a}(\theta_L,\phi_L)] \quad , \tag{2}$$

here, $\boldsymbol{a}(\theta_k,\emptyset_k)$ represents the SV for kth signal. To drive $\boldsymbol{a}(\theta_k,\emptyset_k)$ for arbitrary array geometry shown in Figure (1), the unit vector, $\boldsymbol{u}_k$, containing both θ_k and $\emptyset_k$ needs to be computed as

$$\boldsymbol{u}_k = \cos\,\phi_k \sin\theta_k\,\hat{a}_x + \sin\,\phi_k \sin\theta_k\,\hat{a}_y \; + \; \cos\theta_k\,\hat{a}_z \quad , \tag{3}$$

where, $\hat{a}_x, \hat{a}_y$, and $\hat{a}_z$ represent the unite vectors for Cartesian coordinates. Then, the second unit vector,$\boldsymbol{v}_i$, is required to determine the spatial distance between reference element (i.e., element 1) and the ith element of the array as follows:

$$\boldsymbol{v}_i = r_i cos\vartheta_i \hat{a}_x + r_i sin\vartheta_i\,\hat{a}_y + z_i\hat{a}_z\,, i = 1,2,\cdots,M \;\; , \tag{4}$$

here, ϑ_i represents the angle between x-plane and the position of the ith array element. To procced, the angle between $\boldsymbol{u}_k$ and $\boldsymbol{v}_i$ for the ith sensor with respect to the reference element can be computed as

$$\alpha_{ik} = \cos^{-1}\left(\frac{\boldsymbol{v}_i.\boldsymbol{u}_k}{\|\boldsymbol{v}_i\|.\|\boldsymbol{u}_k\|}\right) = \cos^{-1}\left(\frac{\sin\theta_k\cos(\phi_k-\vartheta_i) + \; z_i\cos\theta_k}{\|\boldsymbol{v}_i\|.\|\boldsymbol{u}_k\|}\right)$$
$$\alpha_{ik} = \cos^{-1}(\sin\theta_k\cos(\phi_k-\vartheta_i)) + \; z_i\cos\theta_k \;\; . \tag{5}$$

An $M \times L$ matrix can be used to represent the whole set of α_{ik}due to collecting L plane waves by M array elements as

$$\boldsymbol{\Gamma}_{ik} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1L} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{M1} & \alpha_{M2} & \cdots & \alpha_{ML} \end{pmatrix}. \tag{6}$$

The time delay, t_{ik}, corresponding to a particular angle, α_{ik}, is computed by

$$\mathrm{t}_{ik} \; = \mathrm{r}\cos\,(\alpha_{ik}) = \mathrm{r}\cos(\cos^{-1}(\sin\theta_k\cos(\phi_k-\vartheta_i)) + \; z_i\cos\theta_k)$$
$$\mathrm{t}_{ik} = r\{\sin\theta_k\cos\;(\phi_k-\vartheta_i) + \; z_i\cos\theta_k\}\,. \tag{7}$$

Similarly, the total set of $ŧ_{ik}$ for L signals received by M antenna is represented by

$$Ŧ_{ik} = \begin{pmatrix} ŧ_{11} & ŧ_{12} & \cdots & ŧ_{1L} \\ ŧ_{21} & ŧ_{22} & \cdots & ŧ_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ ŧ_{M1} & ŧ_{M2} & \cdots & ŧ_{ML} \end{pmatrix}. \quad (8)$$

The angular phase difference, ω_{ik}, can be calculated from the multiplication between $ŧ_{ik}$ and propagation constant, β, as follows:

$$\omega_{ik} = \beta\, ŧ_{ik} = \frac{2\pi}{\lambda}\, ŧ_{ik} \quad , \quad (9)$$

where, λ is the signal wavelength. The total phase differences corresponding to $Ŧ_{ik}$ are presented as

$$\Omega_{ik} = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1L} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{M1} & \omega_{M2} & \cdots & \omega_{ML} \end{pmatrix}. \quad (10)$$

W_{ik} includes information regarding elevation and azimuth angles. Hence, the array SV for kth signal arrived to the array having arbitrary configuration can be computed as follows:

$$\boldsymbol{a}(\theta_k, \theta_k) = \left[\, e^{-j\omega_{1k}},\ e^{-j\omega_{2k}}, \cdots ,\quad e^{-j\omega_{Mk}}\ \right]. \quad (11)$$

3. Principle of Subspace-based DoA Estimation

The measured data matrix for signals incident on the array shown in Figure (1) at N snapshots can be presented by

$$\boldsymbol{X}(t) = \begin{pmatrix} x_1(t_1) & x_1(t_2) & \cdots & \cdots & x_1(t_N) \\ x_2(t_1) & x_2(t_2) & \vdots & \vdots & x_2(t_N) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_M(t_1) & x_M(t_2) & \cdots & \cdots & x_M(t_N) \end{pmatrix}. \quad (12)$$

The true CM, $\boldsymbol{R_{xx}}$, can be computed from $\boldsymbol{X}(t)$ as follows:

$$\boldsymbol{R_{xx}} = E[\boldsymbol{X}(t)\boldsymbol{X}(t)^H] = E[\boldsymbol{AS}(t)\boldsymbol{S}^H(t)\boldsymbol{A}^H] + E[\boldsymbol{N}(t)\boldsymbol{N}^H(t)]$$
$$\boldsymbol{R_{xx}} = \boldsymbol{AR}_{ss}A^H + \sigma_n^2\, \boldsymbol{I}_M \quad , \quad (13)$$

$\boldsymbol{R}_{ss} \in \mathbb{C}^{L\times L} = E[\boldsymbol{s_n}\, \boldsymbol{s_n}^H]$ is the CM of the signal. $\sigma^2 \boldsymbol{I}_{M\times M}$ depicts the noise CM where $\boldsymbol{I}$ is the $M \times M$ identity matrix and σ^2 is the noise variance. $(.)^H$ represents Hermitian transpose operator. The DoAs will vary when the observed sources are non-stationary. This means that the content of $\boldsymbol{A}$ will change partially or completely, which implies that the content of the spatial CM, $\boldsymbol{R_{xx}}$, will also change. Thus, to track the signal characteristics of non-stationary sources and obtain accurate DoA estimation, CM needs to be estimated periodically over N samples as follows:

$$\widehat{\boldsymbol{R}}_{xx} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{X}(t)\boldsymbol{X}^H(t) = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1M} \\ r_{21} & r_{22} & \cdots & r_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ r_{M1} & r_{M2} & \cdots & r_{MM} \end{pmatrix}. \quad (14)$$

The working principle of MUSIC algorithm which is the main representative of the subspace-based method relies on the orthogonality between SV, which includes the information about the direction of the signals and NSS. To achieve this, the observed CM, $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}}$, needs to be decomposed using EVD technique as

$$\widehat{\boldsymbol{R}}_{\boldsymbol{xx}} = \boldsymbol{Q}\,\boldsymbol{\Lambda}\,\boldsymbol{Q}^H = [\boldsymbol{Q_s}\,\boldsymbol{Q_n}] \begin{bmatrix} \boldsymbol{\Lambda_s} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma^2\boldsymbol{I} \end{bmatrix} [\boldsymbol{Q_s}\,\boldsymbol{Q_n}]^H \quad , \qquad (15)$$

here, the columns of $\boldsymbol{Q}$ represent the eigenvectors of $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}}$ and it is partitioned into $\boldsymbol{Q_s} \in \mathbb{C}^{M \times \mathrm{L}}$ and $\boldsymbol{Q_n} \in \mathbb{C}^{M \times M-L}$. $\boldsymbol{Q_s}$ depicts the SSS containing the eigenvectors corresponding to the L-large eigenvalues of $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}}$. $\boldsymbol{Q_n}$ is the NSS containing eigenvectors corresponding to the $M-L$ small eigenvalues of $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}}$. $\boldsymbol{\Lambda}$ is a diagonal-matrix which includes the eigenvalues of $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}}$ and it is partitioned into $\boldsymbol{\Lambda_s}$ and $\sigma^2\boldsymbol{I}$. $\boldsymbol{\Lambda_s} \in \mathbb{R}^{L \times L}$ is a diagonal matrix whose elements are the L large eigenvalues. $\sigma^2\boldsymbol{I} \in \mathbb{R}^{M-L \times M-L}$ is the scaled-identity matrix having $M-L$ small eigenvalues. Subspace-based algorithms use the following formula to generate spatial spectrum:

$$P_{\text{subspace}}(\theta,\phi) = \frac{1}{\| \boldsymbol{Q}_n\, \boldsymbol{a}(\theta,\phi)\|^2} \qquad (16)$$

where $||.||$ denotes norm operator. The peak(s)of this spectrum represent the DoA of the incident signals.

4. Numerical Simulations and Discussion

We investigate the impact of different parameters, such as SNR level, number of antenna elements (M), number of data snapshots (N), spatial distance between antenna elements (d), number of sources (L), and correlation between sources, on the performance of estimated DoA using equation (16). In this work, the resolution of the algorithm is defined as the ability of the method to resolve closely separated sources. Additionally, the accuracy of the DoA estimation is computed based on the Average Root Mean Square Error (ARMSE) and Probability of Successful Detection (PSDs) as follows:

$$\text{ARMSE} = \frac{1}{K}\sum_{i=1}^{K}\sqrt{\frac{1}{L}\sum_{k=1}^{L}\left[\left(\theta_k - \hat{\theta}_k\right)^2\right]} \quad , \qquad (17)$$

$$PSD(DoA) = \frac{\sum_{i=1}^{K} NSD_i}{KL} \quad , \qquad (18)$$

where, K represents the number of Monte Carlo trails, $\theta_k$and $\hat{\theta}_k$ are true and estimation DoA, respectively. NSD_i stands for Number of Successful Detections at ith trial.

4.1. The Impact of SNR Fluctuations

It is well-known that in wireless communications SNR level fluctuates due to the multipath effect and channel fading phenomena. Therefore, we evaluate the impact of SNR variations on resolution and accuracy of the subspace-based DoA estimation technique. To this end, we simulate six plane waves ($L = 6$) impinging on the array from random directions, where the array composite of twelve antenna elements ($M = 12$). It is assumed that SNR level varies from low (-10 dB) to medium (0 dB) and to high (10 dB). The number of snapshots to generate the data matrix is set to be 100 samples. The achieved result is illustrated in Figure (2). Figure (2) shows that when SNR is -10 dB, only four DoAs are detected and two of the sources are missed. When SNR level increases to 0 dB, five DoAs are correctly detected. As expected, in the case of 10 dB SNR, all the six random DoAs are detected with very-high resolution.

Further, we apply equations (17) and (18) to analysis the effect of SNR variation on DoA estimation accuracy. The simulation parameters are the same as the previous scenario and SNR level changes from -10 dB to 10 dB with 5 dB increments. To generate a fair result, five hundred trials ($K = 500$) of three DoAs ($L = 3$) are randomly generated. The ARMSE and PSD are calculated for every SNR level and plotted as shown in Figure (3).

Figure (3) shows that changing SNR values from -10 dB to 10 dB reduces the estimation error from 1.43° to 0.4° respectively. Meanwhile, the PSD has increased from just under 92% to over 97%. Figures (2 and 3) can be interpreted as follows. In case of low SNR, the noise eigenvalues will be very close to the signal eigenvalues. This causes overlapping (poor separation of) the SSS and NSS and introduces difficulty for the algorithm to detect the directions of the sources correctly. On the other hand, noise eigenvalues become too low compared to signal eigenvalues when SNR is high, and the subspaces are well separated. This allows the algorithm to localize the sources with high resolution and minimum error. Therefore, to achieve low estimation error and high successful detections, it is of a significant practical importance to control the SNR level within acceptable range.

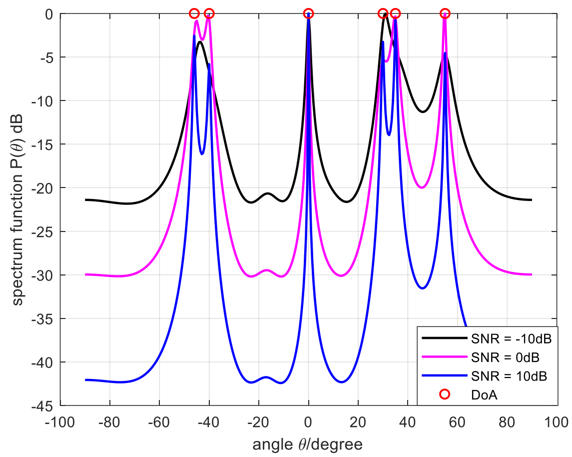
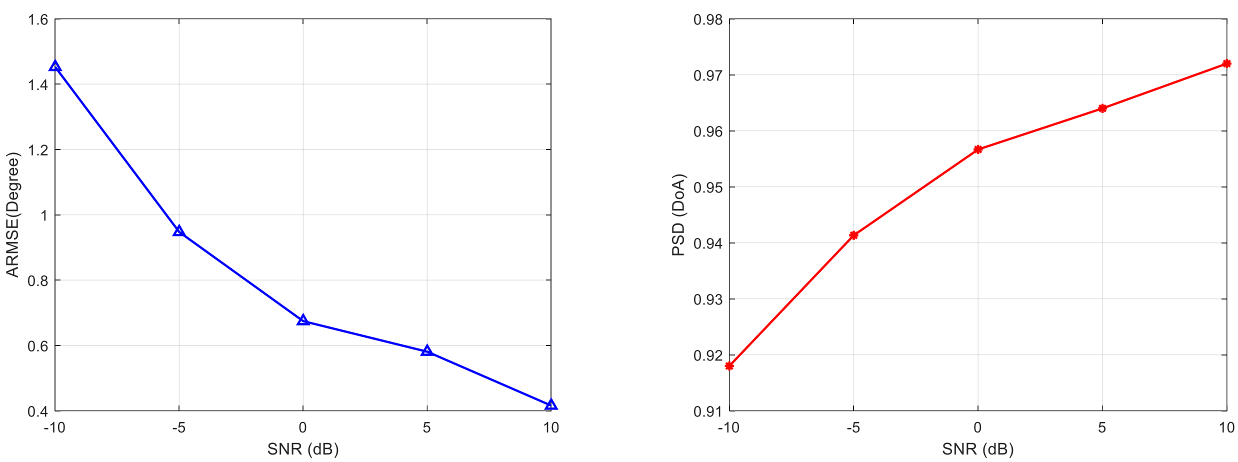


Figure 2. The impact of SNR variations on DoA estimation resolution.



(a) (b)

Figure 3. The Impact of SNR variations on DoA estimation performance. (a) ARMSE vs. SNR, (b) PSD (DoA) vs. SNR.

4.2. The Impact of Number of Snapshots

In this scenario, the impact of number of snapshots on DoA estimation performance is investigated. To this end, it is assumed that five sources ($L = 5$) transmit their signals toward ten-element array ($M = 10$) from random directions. The SNR level is fixed at 0 dB and number of snapshots are changed as $N = [15\ \ 30\ 100]$. The spatial spectrum is plotted for each value of N as shown in Figure (4). Figure (4) depicts that when number of snapshots are $N = 15$ and $N = 30$ the closely separated sources are not resolved completely, only three and four sources are successfully detected, respectively. Contrastingly, in case of high number of snapshots (i.e., $N = 100$) all five DoAs are detected with high-resolution. Therefore, it is observable that increasing N positively impacts the resolution of the DoA estimation and results better noise robustness.

We also observed the variations of ARMSE and PSD with changing the number of available snapshots. The simulation parameters set as follows: $M = 12,\ L = 6,\ \text{SNR} = 0$ dB, and six different number of snapshots, namely $N = [15\ 25\ 50\ 70\ 85\ 100]$ are considered. For each value of N, the program is run five-hundred times ($K = 500$) and the accuracy criteria are calculated and plotted as shown in Figure (5). Figure (5) illustrates that ARMSE reduces gradually from 3.55° to 0.55° as N increases from 15 to 100, respectively. Meanwhile, the PSD is significantly improved with taking more snapshots. Intuitively, both Figures (4 and 5) show that the resolution and accuracy of the subspace-based DoA estimation can be enhanced by increasing the number of snapshots. These results can be interpreted as follows. When the available number of snapshots are not high enough, the arrival signals are not independent. This phenomenon will lead to the so-called rank-deficiency problem in the estimated CM. On other hand, when a high-number of snapshots are available, the CM will obtain full-rank property (i.e., all the eigenvalues will be real and unique). To conclude, higher value of N increases the rank of CM, this ends up with higher estimation resolution and localization accuracy.

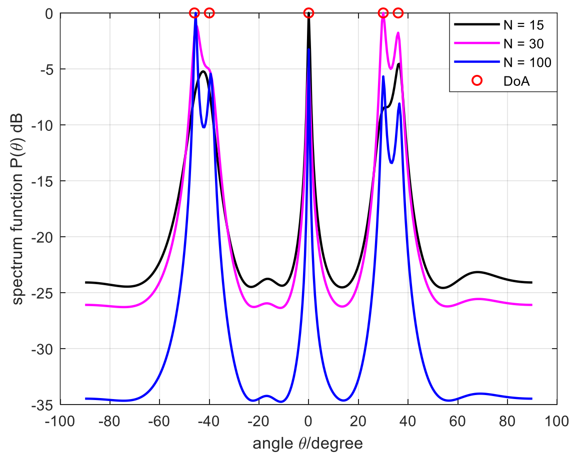
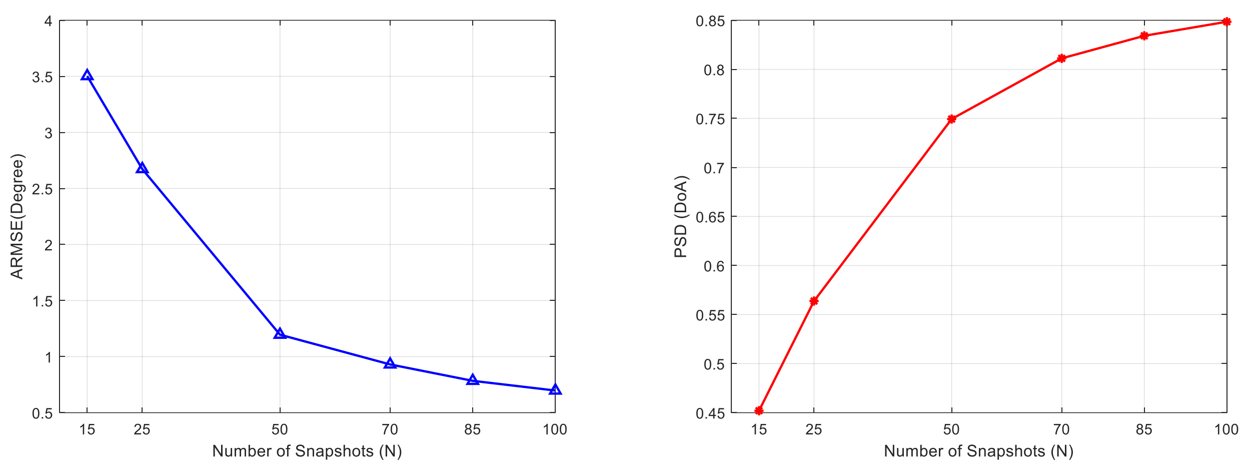


Figure 4. The impact of the number of snapshots on DoA estimation resolution.



(a) (b)

Figure 5. The Impact of number of snapshots on DoA estimation performance. (a) ARMSE vs. Number of snapshots, (b) PSD(DoA) vs. Number of snapshots.

However, the complexity of CM generation is proportional to values of N, i.e., $\mathcal{O}(M^2N)$. Thus, increasing N enlarges the dimension of CM which needs higher number of operations to process as shown in Figure (6). Therefore, the number of snapshots tradesoff between performance and computational complexity in subspace-based DoA estimations.

4.3. The Impact of Number of Array Elements

In this section, we demonstrate how physical array-dimension impacts the DoA estimation performance. The main simulation parameters are adjusted as follows: SNR $= 0$ dB,$N = 100$,$L = 4$, and we change the array elements as $M = [5\ \ 8\ \ 16]$. To evaluate the impact of M on estimation resolution, pseudospectrum for each value of M is computed and the result is shown in Figure (7).

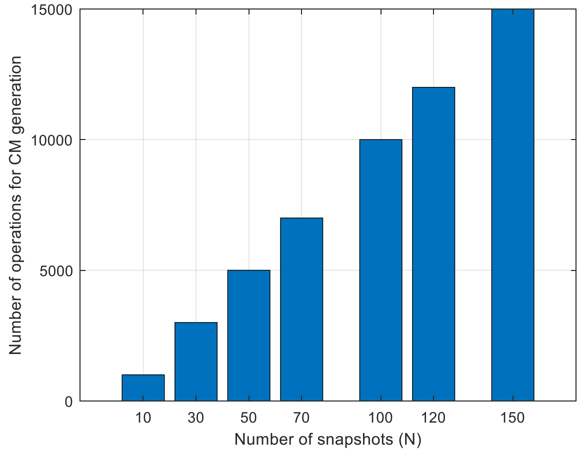


Figure 6. Number of snapshots vs. CM generation complexity.

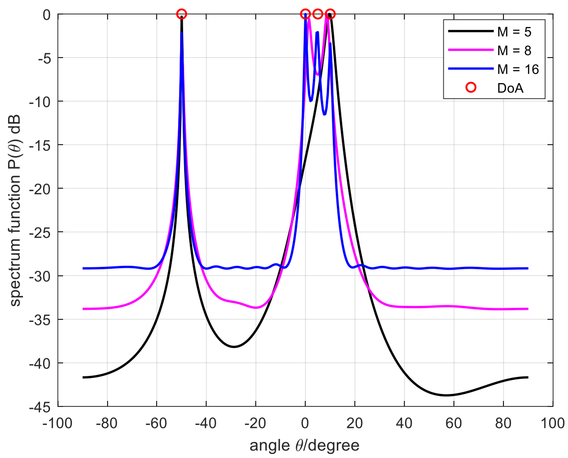
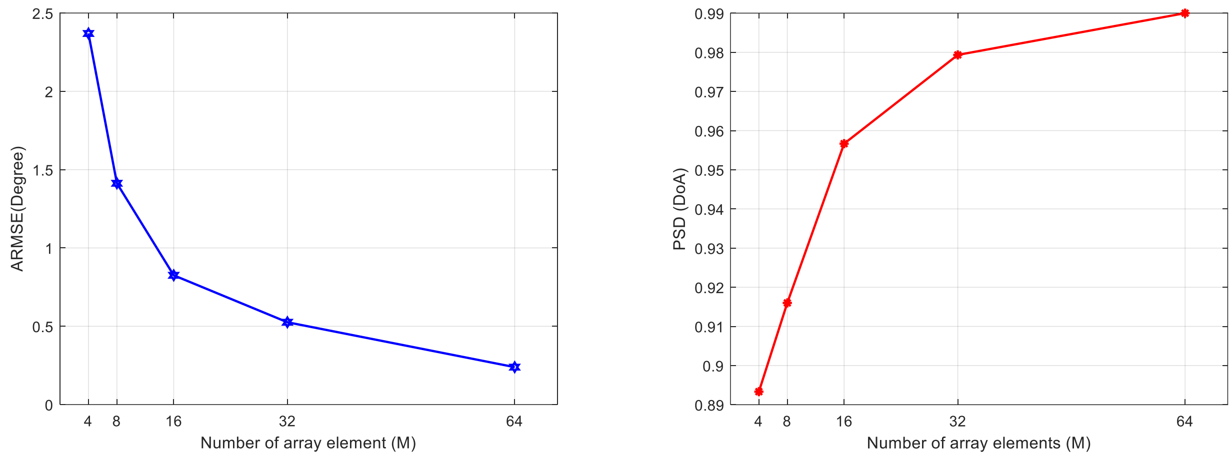


Figure 7. The impact of the number of array elements on DoA estimation resolution.

Understandably, when $M = 5$, only two sources are detected in which three closely separated sources detected as a single source. The number of detected angles rises to three as we use $M = 8$. When $M = 16$, four sharp and clear peaks are generated towards the direction of the all the sources with a high-resolution.

To confirm this result, we observe the accuracy of estimation in terms of estimation error and successful detections with changing the value of $M = [4\ \ 8\ \ 16\ \ 32\ \ 64]$. Two hundred Monte Caro trials are conduced ($K\ = 200$) and the results are averaged and plotted as shown in Figure (8). Figure (8) illustrates that significant improvement in terms of lowering ARMSE and improving PSD are achieved as we increase the number of array elements. This performance enhancement comes from the fact that high value of M increases the array aperture (i.e., DoF) which allows collecting more information about the immitted signals and detect their directions correctly.

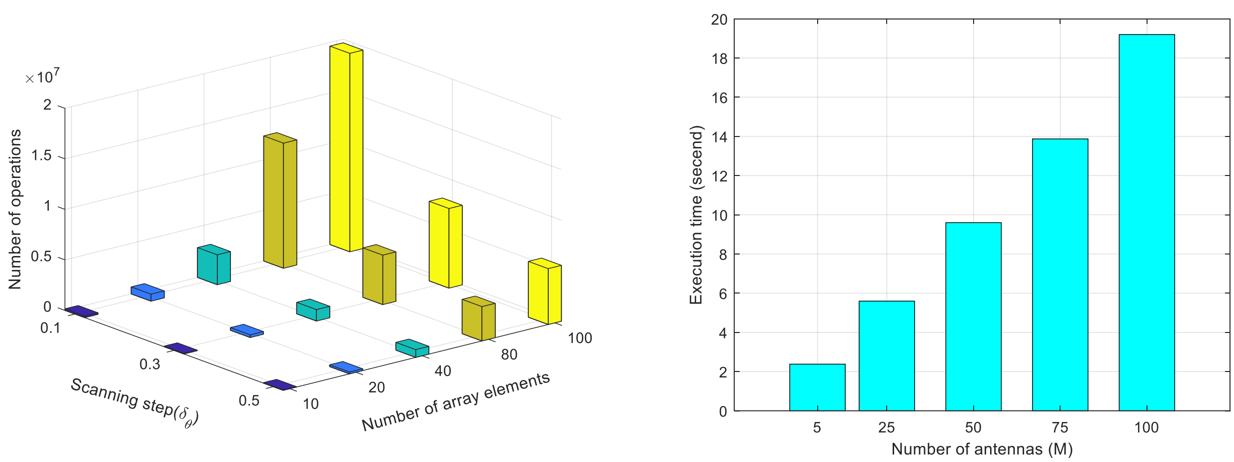
Contrastingly, small antenna array size makes it difficult to collect enough information about the sources. This ends up with higher estimation error and lower PSDs. However, large array size results in increasing computational complexity of DoA algorithms. This is because the number of array elements contributes to the complexity in all three stages adopted in the majority of MUSIC-like DoA estimation techniques, namely CM contraction $\boldsymbol{O}(M^2N)$, CM decomposition $\boldsymbol{O}(M^3)$, and searching grid $\boldsymbol{O}(M^2J_\theta)$. For one direction (1D) DoA estimation, J_θ covers the range of $[-90°\ \ 90°]$. Therefore, $J_\theta = 181/\delta_\theta$. To illustrate the effect of M on the overall complexity, we fix the value of N as 100, and we calculate the total computational burden based on different values of M and various scanning step, δ_θ, as shown in Figure (9a). It is generally accepted that increasing the number of operations leads to higher execution time and ends up with slower algorithm. To illustrate this fact, we also compute the execution time of MUSIC based DoA estimation using previous simulation parameters with different number of array elements, $M\ = [5\ \ 25\ \ 50\ \ 75\ \ 100]$. We use CPU runtime (CPU 2.20 GHz, 4 GB Installed RAM), which corresponds to the needed number of multiplication flops, to calculate the algorithms' time complexity. MATLAB version R2021a is used for running the codes. The program was run one-hundred times ($K = 100$) and the total exclusion time is recorded as show in the Figure (9b).



(a) (b)

Figure 8. The Impact of number of array elements on DoA estimation performance. (a) ARMSE vs. Number of array elements, (b) PSD(DoA) vs. Number of array elements.

Figure (9b) confirms that DoA estimation techniques need more time to converge as the number of antennas increases. Therefore, in massive Multiple Input and Multiple Output (massive-MIMO) systems minimizing the computational complexity due to high-number of antennas is particularly important. Thus, as demonstrated by Figures (7, 8, and 9), tradeoff exists between DoA estimation performance and complexity introduced due to large physical array size.



(a) (b)

Figure 9. The impact of array size on DoA estimation complexity. (a) overall computational complexity vs array elements and different scanning steps, (b) execution time (second) vs number of array elements.

4.4. The Impact of Spatial Distance Between Array Elements

This section tries to identify which spatial separation between adjacent array elements gives best estimation performance. To this end, we simulate a scenario where three sources ($L = 3$) send random signals from unknown directions towards eight elements ($M = 8$) ULA. Other simulation parameters are SNR $= 0$ dB and $N = 100$. We considered four different spatial separations, $d = [0.2\lambda \;\; 0.5\lambda \;\; 1\lambda \;\; 1.2\lambda]$. To analysis the estimation resolution, we generate the spatial spectrum for each value of d as shown in Figure (10).

Figure (10) illustrates that when $d = 0.2\lambda$ (i.e., the array elements are close to each other), two wide peaks are generated, where one of them totally deviated from the actual direction This is due to severe self-interference between array elements known as Mutual Coupling (MC). This interference causes unwanted addition of phase and magnitude errors, and this leads to poor estimation performance of the direction-finding algorithms. When the value of d is large (i.e., $d = 1\lambda$, $1.2\ \lambda$), even though the directions of all the sources are detected, multiple maxima known as grading lobes, are generated in other directions. When $d = 1\lambda$ and 1.2λ the number of grading lobes are four and five, respectively.

Therefore, by increasing d the number of grading lobes also increases. On the other hand, when $d = 0.5\ \lambda$ the directions of all the sources are detected clearly without generating any picks in wrong directions. To sum up, this result confirms that 0.5λ is the optimum choice for spatial separation between array elements for DoA estimation and provides best resolution.

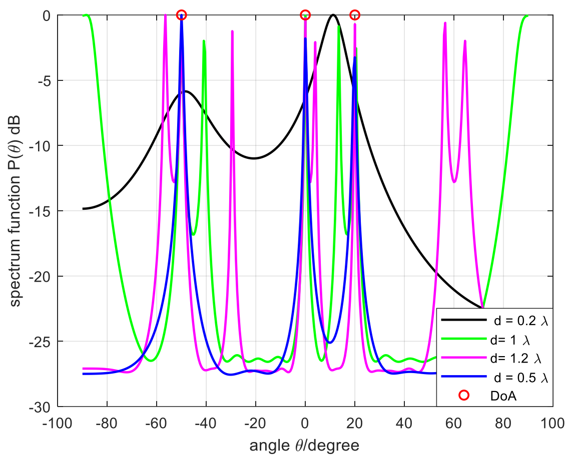


Figure 10. The impact of spatial separation between array elements on estimation resolution.

To further confirm the above result, we also calculate ARMSE with changing the value of d as shown in Figure (11). The main simulation parameters are set as follows: SNR $= 0$ dB, $M = 12$, $L = 3$, $N = 100$, $K = 200$. As expected, the minimum error occurs in the DoA estimation when $d = 0.5\ \lambda$. This result can be interpreted as follows. When the inter-element spacing in array antenna is less than $0.5\ \lambda$, the MC between the elements is severe, which results invalid initial calibration of the SV. Additionally, small value of d makes the collected signals correlated. These cause more errors in the DoA estimation. When d is greater than $0.5\ \lambda$, ARMSE rises sharply due to generating large picks in wrong directions as shown in Figure (10). Therefore, in ULA configuration, $d = 0.5\ \lambda$ is the best choice for high resolution and low ARMSE in DoA estimation. In many critical applications, it is required to compact the overall array size. One possible way to achieve this could be by reducing the value of d ($d < 0.5\ \lambda$). In this situation, decoupling technique is needed to compensate for the MC effect.

4.5. The Impact of Available Number of Sources

The objective in this scenario is to investigate the effect of the number of available sources on DoA estimation performance. To this end, a fixed-size array with $M = 10$ is used and the number of far-field sources are varied such that $L = [2\ \ 5\ \ 8]$. To make a fair comparison, the angular separation between the two adjacent sources is fixed at 10°. Other simulation parameters are SNR$= 5$ dB and $N = 100$. The obtained result is shown in Figure (12).

The result depicts that when $L = 2$, the directions of both sources are detected clearly with high-resolution. However, the estimation resolution is significantly impacted when L increases. Such that, when $L = 5$ only four of the sources are detected.

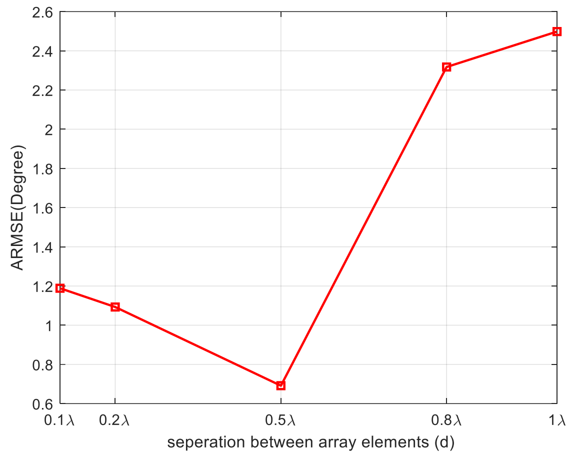


Figure 11. Illustration of optimum inter-element spacing of the array in terms of estimation error.

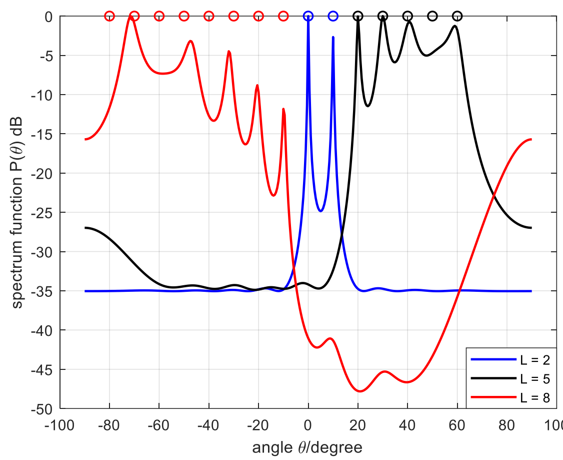


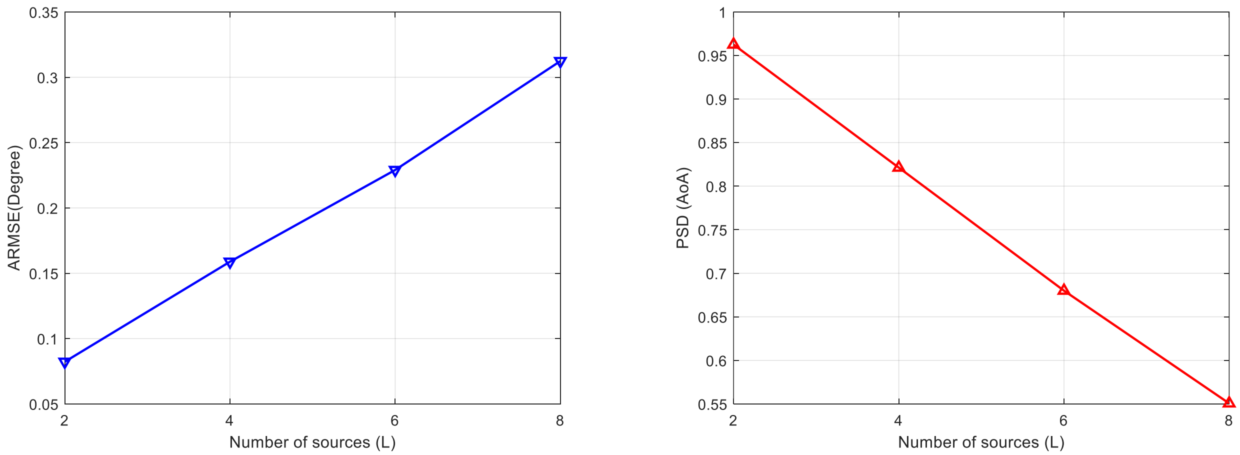
Figure 12. The impact of the number of sources on DoA estimation resolution.

Unsurprisingly, when $L = 8$ only five sources are detected and three of them are missed during the detection process. The reason for that is an antenna array with fixed elements has a certain Field of View (FoV) and in case of high number of sources, some of the sources may not be within the array FoV region, therefore, they are not correctly detected. Our result reflects the fact that for unambiguous detection, L must be less than $M/2$.

The impact of the number of sources on estimation accuracy is also evaluated. We assume four cases where $L = [2\ \ 4\ \ 6\ 8]$. The program is run two-hundred times ($K = 200$) and the accuracy criteria for individual value of L is calculated as shown in Figure (13). Noticeably, it can be observed that the estimation error rises exponentially with increasing L . Predictably, the PSD decreases gradually as more sources are involved in the detection process. These results confirm the previously shown result that fixed-element array (i.e., fixed array aperture) has ability to detect a limited number of sources successfully.

4.6. The Impact of Correlation Between Sources

The impact of correlation or dependency between the sources on the performance of direction estimation systems is investigated in this section. To demonstrate that, we estimate DoAs for two correlated sources. To clearly observe the effect, we assume three different dependency levels between the incident signals such that Correlation Coefficient (CC) = [0.55 0.85 0.95]. For each value of CC, we plot the estimated peaks as shown in Figure (14). When CC =0.55, the algorithm still can resolve two sources. When CC$= 0.85$, only one peak is generated (totally deviated from actual angles). When the two sources are highly correlated (i.e., CC $= 0.95$) none of the angles are correctly detected.



(a) (b)

Figure 13. The impact of the number of sources on DoA estimation accuracy. (a) ARMSE vs. number of sources, (b) PSD (DoA) vs. number of sources.

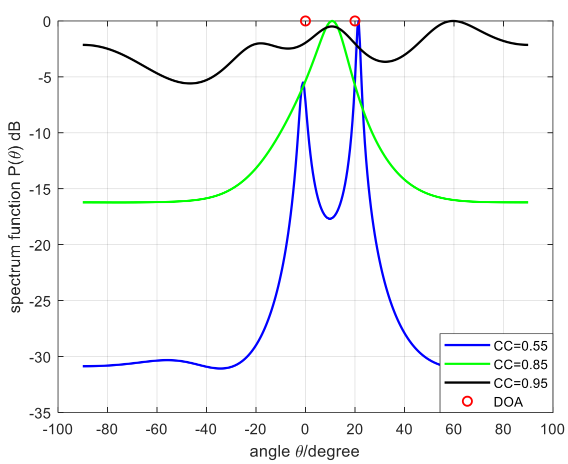


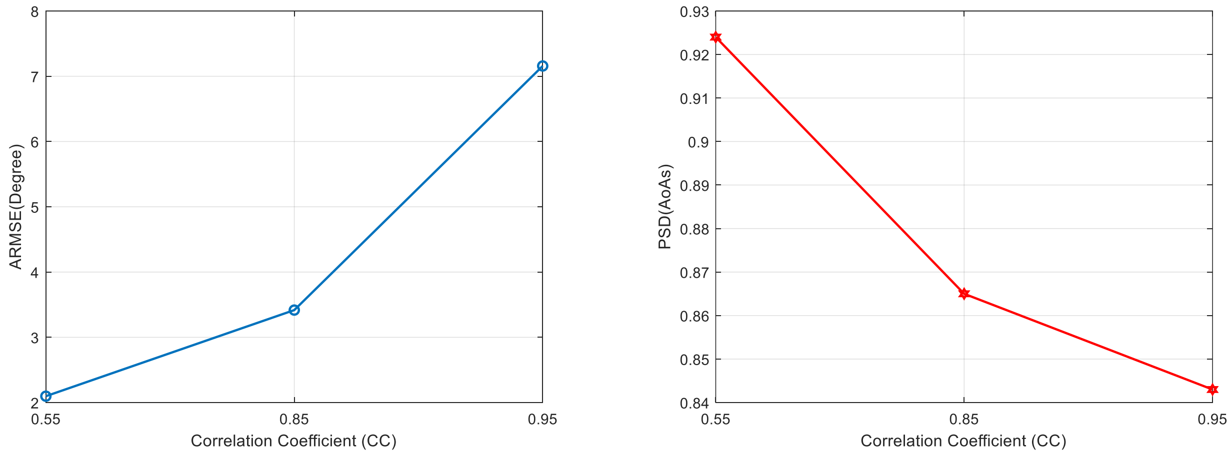
Figure 14. The impact of the correlation between sources on DoA estimation resolution.

Using the same simulation parameters, the effect of dependency between the sources on estimation accuracy is also analyzed and the result is illustrated in Figure (15). Expectably, Figure (15) shows that DoA estimation performance is negatively impacted as sources become more correlated. Thus, MUSIC-like algorithms cannot resolve correlated signals due to the interaction between NSS and SSS. To solve this problem, pre-processing, such as SS techniques, are required to decorrelate the signals before inserting into the subspace-based DoA algorithms.

5. Conclusion

In this work, the impact of SNR, number of array elements, available snapshots, distance between array elements, number of sources, and correlation between sources on the performance of subspace-based DoA estimation has been analyzed. Several numerical simulations have been conducted to evaluate the impact of these parameters on resolution, accuracy, and complexity of MUSIC-like DoA estimation method. It is found that the estimation performance can be improved by lowering the noise level (high SNR) and/or taking more snapshots to generate data matrix at the output of the array and/or using large antenna array. However, tradeoff between performance and computational complexity needs to be considered as the array size and/or the number of snapshots is increased. The results have demonstrated that to minimize the MC effect and avoiding grading lobes, $0.5\,\lambda$ is the best choice for separation between array elements. That is to say, if $d < 0.5\,\lambda$ and $d > 0.5\,\lambda$ the DoA estimation performance is negatively affected due to MC and grading lobes, respectively. It has been also shown that for unambiguous detection, number of detectable sources must be less than

$M/2$. Finally, the results have confirmed that subspace-based DoA estimation algorithms are highly sensitive to correlated signals. Pre-processing technique such as SS can be applied to overcome this problem at the cost of adding complexity. In the feature work, this analysis will be used as fundamental facts for developing more robust subspace-based DoA estimation techniques. Additionally, it will help to select optimum tradeoff between complexity and performance based on applied applications.



(a) (b)

Figure 15. the impact of correlated sources on DoA estimation performance. (a) ARMSE vs. CC, (b) PSD(DoA) vs. CC.

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